Theory of restriction degree of Triple I method with total inference rules of fuzzy reasoning*

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Received March 27, 2000; revised April 17, 2000

Abstract Based on the theory of sustentation degree of Triple I method together with the formulas of α-Triple I modus ponens (MP) and α-Triple I modus tollens(MT), the theory of restriction degree of Triple I method is proposed. Its properties are analyzed, and the general formulas of supremum of α-Triple I MP and infimum of α-Triple I MT are obtained.

Keywords: fuzzy reasoning, restriction degree, α-Triple I method.

Based on Refs. [1—4], the generalized form of optimization problem for the theory of sustentation degree of Triple I method is proposed in Ref. [5], which is as follows.

For any $\alpha \in [0,1]$ with known $A \in \mathbb{F}(X), B \in \mathbb{F}(Y)$ and $A^* \in \mathbb{F}(X)$ (or $B^* \in \mathbb{F}(Y)$), we seek for the optimal $B^* \in \mathbb{F}(Y)$ (or $A^* \in \mathbb{F}(X)$) satisfying

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \geqslant \alpha$$

for any $x \in X$ and $y \in Y$, where $\mathbb{F}(X)$ and $\mathbb{F}(Y)$ represent the families consisting of all fuzzy subsets of X and Y respectively.

However, when the fuzzy reasoning is applied to fuzzy control system, we have to investigate the inverse problem, i.e. to seek the optimal $B^* \in \mathbb{F}(Y)$ (or $A^* \in \mathbb{F}(X)$) such that

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \leqslant \alpha. \tag{1}$$

In this paper, the above inverse problem is discussed by using the implication operator R_0 given in Ref. [5]. The theory of restriction degree of Triple I method is proposed, and the computation formulas for supremum of α-Triple I MP and infimum of α-Triple I MT are given. The theory of α-Triple I method is further developed. In addition, this work provides a necessary theoretical foundation for realizing some performance index of a new type of fuzzy controllers.

1 Theory of restriction degree

For convenience, if the symbols of this paper are not explained specially, their meanings

* Project supported by the National Natural Science Foundation of China (Grant No. 69934010).

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are the same as in Ref. [5]. Some definitions are as follows.

(i) FMP (fuzzy modus ponens). Suppose that X and Y are nonempty sets, $A, A^* \in \mathbb{F}(X)$, $B \in \mathbb{F}(Y)$, then the fuzzy set $B^* \in \mathbb{F}(Y)$ satisfying (1) is called an α-solution of (1) for Triple I FMP.

(ii) FMT (fuzzy modus tollens). Suppose that X and Y are nonempty sets, $A \in \mathbb{F}(X)$, B, $B^* \in \mathbb{F}(Y)$, then the fuzzy set $A^* \in \mathbb{F}(X)$ satisfying (1) is called an α-solution of (1) for Triple I FMT.

(iii) Restriction degree. Suppose that Z is a nonempty set, $\alpha \in [0,1]$, C, $D \in \mathbb{F}(Z)$. If $\sup\{C(z) \to D(z) \mid z \in Z\} = \alpha$, then the restriction degree of C relative to D is said to be α, and is denoted by $\mathrm{rest}(C, D) = \alpha$.

Obviously, (1) holds if and only if the restriction degree of $A \to B$ relative to $A^* \to B^*$ is less than or equal to α. When the implication operator R_0 is adopted, then for $\alpha \in (0, 1)$, $\mathrm{rest}(C, D) \leqslant \alpha$ if and only if $C(z) > D(z)$, $C'(z) \leqslant \alpha$ and $D(z) \leqslant \alpha$ whenever $z \in Z$, where $C'(z) = 1 - C(z)$. The general properties of the restriction degree are as follows.

Theorem 1. Suppose that $\mathrm{rest}(A, B) = \alpha < 1$, $\mathrm{rest}(B, C) = \beta < 1$, then $\mathrm{rest}(A, C) \leqslant \alpha \wedge \beta$ and $\alpha + \beta \geqslant 1$.

Proof. By the hypothesis that $\mathrm{rest}(A, B) = \alpha < 1$, it is known that

$$A(z) > B(z). \tag{2}$$

From $R_0(A(z), B(z)) = A'(z) \vee B(z) \leqslant \alpha$, we get

$$A'(z) \leqslant \alpha \text{ and } B(z) \leqslant \alpha \tag{3}$$

whenever $z \in Z$. In a similar way, from $\mathrm{rest}(B, C) = \beta < 1$, we know

$$B(z) > C(z), \tag{4}$$

and from $R_0(B(z), C(z)) = B'(z) \vee C(z) \leqslant \beta$, we have

$$B'(z) \leqslant \beta \text{ and } C(z) \leqslant \beta \tag{5}$$

whenever $z \in Z$. Using (2) and (4), we have $A(z) > B(z) > C(z)$, so that

$$R_0(A(z), C(z)) = A'(z) \vee C(z). \tag{6}$$

Furthermore, from $A'(z) < B'(z) < C'(z)$ and (3), we have $A'(z) \leqslant \alpha \wedge \beta$, and using (3)—(5), it follows that $C(z) \leqslant \alpha \wedge \beta$. Consequently, by (6) again, we obtain

$$R_0(A(z), C(z)) \leqslant \alpha \wedge \beta$$

whenever $z \in Z$. Hence

$$\mathrm{rest}(A, C) = \sup\{R_0(A(z), C(z)) \mid z \in Z\} \leqslant \alpha \wedge \beta.$$

And noting that $B'(z) \leqslant \beta$ and $B(z) \leqslant \alpha$, it follows that $\alpha + \beta \geqslant 1$. Q.E.D.

Suppose that A, B, A_i and $B_i \in \mathscr{F}(Z)(i \in I)$, then the following expressions are valid:

(i) $\mathrm{rest}(\bigwedge_{i \in I} A_i, B) = \bigvee_{i \in I} \mathrm{rest}(A_i, B)$,

(ii) $\mathrm{rest}(A, \bigvee_{i \in I} B_i) = \bigvee_{i \in I} \mathrm{rest}(A, B_i)$.

Suppose that A, B and $C \in \mathscr{F}(Z)$, then

(iii) $\mathrm{rest}(A, B \to C) = \mathrm{rest}(B, A \to C)$,

(iv) $\mathrm{rest}(A, B \to C) = \mathrm{rest}(A, C' \to B')$.

Suppose that A, B, C, B_i and $C_i \in \mathscr{F}(Z)$, $(i \in I)$, then

(v) $\mathrm{rest}(A, \bigwedge_{i \in I} B_i \to C) = \bigvee_{i \in I} \mathrm{rest}(A, B_i \to C)$,

(vi) $\mathrm{rest}(A, B \to \bigvee_{i \in I} C_i) = \bigvee_{i \in I} \mathrm{rest}(A, B \to C_i)$.

2 Supremum (infimum) of α-Triple I FMP (FMT) for R_0

Concerning the theory of restriction degree, when $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(Y)$ and $A^* \in \mathscr{F}(X)$ (or $B^* \in \mathscr{F}(Y)$) are known, there may be no solution B^* (or A^*) satisfying (1), so there may be no optimal solution.

Example 1. Suppose $X = Y = [0, 1]$, $A(x) \equiv 0.7$, $B(y) \equiv 0.6$, $A^*(x) \equiv 0.1$ (or $B^*(y) \equiv 0.9$), then for any $\alpha \in (0, 1)$ there will not exist $B^* \in \mathscr{F}(Y)$ (or $A^* \in \mathscr{F}(X)$) satisfying (1). In fact, by the definition of implication operator R_0, we know

$$R_0(A, B) \equiv 0.3 \vee 0.6 = 0.6.$$

For known $A^*(x) \equiv 0.1$ and any $B^* \in \mathscr{F}(Y)$, we have $R_0(A^*, B^*) \geqslant (A^*)' \equiv 0.9 > 0.6$, so that $R_0(A, B) \to R_0(A^*, B^*) \equiv 1 > \alpha$, i.e. (1) does not hold. For known $B^*(y) \equiv 0.9$ and any $A^* \in \mathscr{F}(X)$, we have $R_0(A^*, B^*) \geqslant B^* \equiv 0.9$. In a similar way, we know that (1) does not hold. However there exists a solution of (1) in a certain condition. Now, we give the following results.

Theorem 2 (supremum of α-Triple I FMP for R_0). Suppose that X, Y are nonempty sets, A, $A^* \in \mathscr{F}(X)$, $B \in \mathscr{F}(Y)$, $0 < \alpha < 1$, then there exists the α-solution of (1) for Triple I FMP with total inference rules if and only if there exists $x_0 \in X$ such that $A^*(x_0) > 0$ and

$$(A^*(x))' < R_0(A(x), B(y)),\ A^*(x) \wedge R_0(A(x), B(y)) \geqslant \alpha' \tag{7}$$

whenever $x \in X$ amd $y \in Y$. On this condition, the supremum of α-solution of (1) for Triple I FMP is determined by

$$B^*(y) = \inf_{x\in X}[A^*(x) \wedge R_0(A(x),\ B(y))] \wedge \alpha,\ y \in Y. \tag{8}$$

Proof (necessity). Suppose that $C(y)$ is an α-solution of (1) for Triple I FMP with total inference rules, i.e.

$$(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow C(y)) \leqslant \alpha \tag{9}$$

whenever $x \in X$ and $y \in Y$. By the definition of R_0, there exists $x_0 \in X$ such that $A^*(x_0) > 0$. Otherwise, $A^*(x) \equiv 0$ implies that $R_0(A^*(x),\ B(y)) \equiv 1$. This contradicts (9). From (9), we can also get

$$A^*(x) > C(y) \text{ and } R_0(A(x),\ B(y)) > R_0(A^*(x),\ C(y)) = (A^*(x))' \vee C(y). \tag{10}$$

Consequently, $(A^*(x))' < R_0(A(x),\ B(y))$. Furthermore, we have

$$\begin{aligned} &(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow C(y)) \\ &= R'_0(A(x),\ B(y)) \vee (A^*(x))' \vee C(y) \\ &= (A^*(x) \wedge R_0(A(x),\ B(y))' \vee C(y) \leqslant \alpha. \end{aligned} \tag{11}$$

Combining (10) and (11), we obtain (7).

Proof (sufficiency). From the proof of necessity of this theorem, it can be seen that by taking $C(y) \equiv 0$ and from the hypothesis in (7), we know that $C(y)$ is an α-solution of (1) for Triple I FMP with total inference rules.

Now we prove that $B^*(y)$ determined by (8) is the supremum of α-solution of (1) for Triple I FMP with total inference rules. In fact, for any $C(y) \in \mathcal{F}(Y)$ and $C(y) < B^*(y)$, we have

$$C(y) < A^*(x),\ C(y) < R_0(A(x),\ B(y)) \text{ and } C(y) < \alpha \tag{12}$$

whenever $x \in X$ and $y \in Y$. So that $R_0(A^*(x),\ C(y)) = (A^*(x))' \vee C(y)$. By the hypothesis in (7) again, and similar to the proof of (11), we can also get

$$(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow C(y)) \leqslant \alpha.$$

i.e. $C(y)$ is an α-solution of (1) for Triple I FMP with total inference rules.

In the following, we prove that for any $D(y) \in \mathcal{F}(Y)$ with $D(y_0) > B^*(y_0)$ where $y_0 \in Y$, $D(y)$ is not the α-solution of (1) for Triple I FMP with total inference rules. In fact, by $D(y_0) > B^*(y_0)$ and the sense of $B^*(y)$ determined by (8), there exists $x_0 \in X$ such that

$$D(y_0) > A^*(x_0) \wedge R_0(A(x_0), B(y_0)) \wedge \alpha. \tag{13}$$

Now let us verify

$$R_0(A(x_0), B(y_0)) \to R_0(A^*(x_0), D(y_0)) > \alpha \tag{14}$$

in different cases.

(Ⅰ) If $A^*(x_0) \leqslant D(y_0)$, then by the definition of R_0, we know $R_0(A^*(x_0), D(y_0)) = 1$; therefore (14) holds.

(Ⅱ) If $A^*(x_0) > D(y_0)$, then

$$R_0(A(x_0), B(y_0)) \to R_0(A^*(x_0), D(y_0))$$

$$= R_0(A(x_0), B(y_0)) \to (A^*(x_0))' \vee D(y_0). \tag{15}$$

Furthermore, we will discuss two cases.

(i) If $R_0(A(x_0), B(y_0)) \leqslant D(y_0)$, by the definition of R_0, (14) holds.

(ii) If $R_0(A(x_0), B(y_0)) > D(y_0)$, and noting that $A^*(x_0) > D(y_0)$, from (13) it can be deduced that $\alpha < D(y_0)$. By the hypothesis in (7) again, we have $(A^*(x_0))' < R_0(A(x_0), B(y_0))$. So

$$R_0(A(x_0), B(y_0)) > (A^*(x_0))' \vee D(y_0).$$

Furthermore we have

$$R_0(A(x_0), B(y_0)) \to R_0(A^*(x_0), D(y_0))$$

$$= R'_0(A(x_0), B(y_0)) \vee (A^*(x_0))' \vee D(y_0) > \alpha.$$

All of these show that (14) holds. Q.E.D.

Under the hypothesis and condition of Theorem 2, for any $y \in Y$, if x is confined to $X_0 = \left\{x \in X \mid \inf_{x \in Y}[A^*(x) \wedge R_0(A(x), B(y))] < A^*(x) \wedge R_0(A(x), B(y))\right\}$, then $B^*(y)$ determined by (8) is an α-solution of (1) for Triple I FMP with total inference rules, it is also the α-maximum solution of (1) for Triple I FMP. If $y \in Y$, x is confined to $X - X_0$, then $B^*(y)$ determined by (8) may not be the α-soloution of (1) for Triple I FMP, in this case, there may not exist α-maximum solution of (1) for Triple I FMP. Furthermore, we give the following discrimination criterion.

Under the hypothesis and condition of Theorem 2, there exists the α-maximum solution of (1) for Triple I FMP with total inference rules if and only if

$$A^*(x) \wedge R_0(A(x), B(y)) > \alpha \text{ whenever } x \notin X_0.$$

On this condition, $B^*(y)$ determined by (8) is the α-maximum solution of (1) for Triple I FMP.

Theorem 3 (infimum of α-Triple I FMT for R_0). Suppose that X, Y are nonempty sets, $A \in \mathbb{F}(X)$, B, $B^* \in \mathbb{F}(Y)$, $0<\alpha<1$, then there exists α-solution of (1) for Triple I FMt with total inference rules if and only if there exists $y_0 \in X$ such that $B^*(y_0)<1$ and

$$B^*(y) < R_0(A(x),\ B(y)),\quad B^*(y) \vee R'_0(A(x),\quad B(y)) \leqslant \alpha \tag{16}$$

whenever x∈X and y∈Y. On this conditon, the infimum of α-solution of (1) for Triple I FMT is determined by

$$A^*(x) = \sup_{y \in Y}[B^*(y) \vee R'_0(A(x),\ B(y))] \vee \alpha',\quad x \in X. \tag{17}$$

Proof (necessity). Suppose that $C(x)$ is an α-solution of (1) for Triple I FMT with total inference rules, i.e.

$$(A(x) \to B(y)) \to (C(x) \to B^*(y)) \leqslant \alpha \tag{18}$$

whenever $x \in X$ and $y \in Y$. By the definition of R_0, there exists $y_0 \in Y$ such that $B^*(y_0)<1$. Otherwise from $B^*(y) \equiv 1$, it follows that $R_0(C(x),\ B^*(y)) \equiv 1$. This contradicts (18). From (18), we have

$$C(x) > B^*(y) \text{ and } R_0(A(x),\ B(y)) > R_0(C(x),\ B^*(y)) = C'(x) \vee B^*(y). \tag{19}$$

Furthermore, we get

$$\begin{aligned}&(A(x) \to B(y)) \to (C(x) \to B^*(y)) \\ &= R'_0(A(x),\ B(y)) \vee C'(x) \vee B^*(y) \leqslant \alpha.\end{aligned} \tag{20}$$

Combining (19) and (20), we know that (16) holds.

Proof (sufficiency). From the proof of necessity, taking $C(x) \equiv 1$, from the hypothesis in (16), it is easy to know that $C(x)$ is an α-solution of (1) for Triple I FMT with total inference rules.

Now we verify that $A^*(x)$ determined by (17) is the infimum of α-solution of (1) for Triple I FMT with total inference rules. In fact, for any $C(x) \in \mathbb{F}(X)$ with $C(x) > A^*(x)$ where $x \in X$, we have

$$C(y) > B^*(y),\ C(y) > R'_0(A(x),\ B(y)),\ C(x) > \alpha' \tag{21}$$

whenever $x \in X$, $y \in Y$. So $R_0(C(x),\ B^*(y)) = C'(x) \vee B^*(y)$. By the hypothesis in (16), we get $R_0(A(x),\ B(y)) > C'(x) \vee B^*(y)$. Furthermore, we have

$$(A(x) \to B(y)) \to (C(x) \to B^*(y))$$

$$= R'_0(A(x), B(y)) \vee C'(x) \vee B^*(y) \leqslant \alpha,$$

i.e. $C(x)$ is an α-solution of (1) for Triple I FMP with total inference rules. In the following, we show that for any $D(x) \in \mathbb{F}(X)$ with $D(x_0) < A^*(x_0)$ where $x_0 \in X$, $D(x)$ is not the α-solution of (1) for Triple I FMT with total inference rules. In fact, from $D(x_0) < A^*(x_0)$ and the sense of $A^*(x)$ determined by (17), there exists $y_0 \in Y$ such that

$$D(x_0) < B^*(y_0) \vee R'_0(A(x_0), B(y_0)) \vee \alpha'. \tag{22}$$

In the following, we will verify

$$R_0(A(x_0), B(y_0)) \to R_0(D(x_0), B^*(y_0)) > \alpha \tag{23}$$

in different cases:

(I) If $D(x_0) \leqslant B^*(y_0)$, by the definition of R_0, we know $R_0(D(x_0), B^*(y_0)) = 1$, hence (23) holds.

(II) If $D(x_0) > B^*(y_0)$, $R_0(D(x_0), B^*(y_0)) = D'(x_0) \vee B^*(y_0)$. we will discuss two cases.

(i) If $R_0(A(x_0), B(y_0)) \leqslant D'(x_0)$, by the definition of R_0, (23) holds.

(ii) If $R_0(A(x_0), B(y_0)) > D'(x_0)$, that is, $R'_0(A(x_0), B(y_0)) < D(x_0)$, from $B^*(y_0) < D(x_0)$ and (22), it follows that $\alpha' > D(x_0)$. Hence $D'(x_0) > \alpha$. By the hypothesis in (16) again, we know $B^*(y_0) < R_0(A(x_0, B(y_0))$. So

$$R_0(A(x_0), B(y_0)) > D'(x_0) \vee B^*(y_0). \tag{24}$$

Furthermore, we have

$$R_0(A(x_0), B(y_0)) \to R_0(D(x_0), B^*(y_0))$$

$$= R'_0(A(x_0), B(y_0)) \vee D'(x_0) \vee B^*(y_0) > \alpha.$$

All of these show that (23) holds. Q.E.D.

On the hypothesis and condition of Theorem 3, for any $x \in X$, if y is confined to $Y_0 = \left\{ y \in Y \mid \sup\limits_{y \in Y} [B^*(y) \vee R'_0(A(x), B(y))] > B^*(y) \vee R'_0(A(x), B(y)) \right\}$, then $A^*(x)$ determined by (17) is an α-solution of (1) for Triple I FMT with total inference rules, and it is also the α-minimum solution of (1) for Triple I FMT. If $x \in X$, y is confined to $Y - Y_0$, then $A^*(x)$ determined by (17) may not be the α-solution of (1) for Triple I FMT. In this case, there may not exist the α-minimum solution of (1) for Triple I FMT. And we obtain the following discriminate criterion.

As the hypothesis and condition of Theorem 3, there exists the α-minimum solution of (1) for Triple I FMT with total inference rules if and only if

$$B^*(y) \vee R'_0(A(x), B(y)) < \alpha' \text{ whenever } y \notin Y_0.$$

In this condition, $A^*(x)$ determined by (17) is the α-minimum solution of (1) for Triple I FMT.

Finally, we consider the following form of generalized problem of the theory of restriction degree of Triple I method with total inference rules, i.e. to seek the optimal B^* (or A^*) such that

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) < \alpha. \tag{25}$$

Then we have following theorems.

Theorem 4 (supremum of α-Triple I FMT for R_0). Suppose that X, Y are nonempty sets, $A^* \in \mathbb{F}(X)$, $B \in \mathbb{F}(Y)$, $0 < \alpha \leqslant 1$, then there exists α-solution of (25) for Triple I FMP with total inference rules if and only if threre exists $x_0 \in X$ such that $A^*(x_0) > 0$ and

$$(A'(x))' < R_0(A(x), B(y)),\ A^*(x) \wedge R_0(A(x), B(y)) > \alpha' \tag{26}$$

whenever $x \in X$ and $y \in Y$. On this condition, the supremum of α-solution of (25) for Triple I FMP is determined by

$$B^*(y) = \inf_{x \in X}[A^*(x) \wedge R_0(A(x), B(y))] \wedge \alpha, \qquad y \in Y. \tag{27}$$

And for any $y \in Y$, if $x \in X_0$, $B^*(y)$ determined by (27) is the α-maximum solution of (25) for Triple I FMP. If $y \in Y$, $x \notin X_0$ then $B^*(y)$ determined by (27) is not the α-solution of (25) for Triple I FMP. In this case, there will not exist the α-maximum solution of (25) for Triple I FMP.

Theorem 5 (infimum of α-Triple I FMT for R_0). Suppose that X, Y are nonempty sets, $A \in \mathbb{F}(X)$, B, $B^* \in \mathbb{F}(Y)$, $0 < \alpha \leqslant 1$, then there exists the α-solution of (25) for Triple I FMT with total inference rules if and only if there exists $y_0 \in X$ such that $B^*(y_0) < 1$ and

$$B^*(y) < R_0(A(x), B(y)),\ B^*(y) \vee R'_0(A(x), B(y)) < \alpha \tag{28}$$

whenever $x \in X$ and $y \in Y$. On this condition, the infimum of α-solution of (25) for Triple I FMT is determined by

$$A^*(x) = \sup_{y \in Y}[B^*(y) \vee R'_0(A(x), B(y))] \vee \alpha', \qquad x \in X. \tag{29}$$

And for any $x \in X$, if $y \in Y_0$, $A^*(x)$ determined by (29) is the α-minimum solution of (25) for Triple I FMT. If $x \in X$, $y \notin Y_0$, then $A^*(x)$ determined by (29) is not the α-solution of (25) for Triple I FMT. In this case, there will not exist the α-minimum solution of (25) for Triple I FMT.

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